Math 122 Friday, October 28

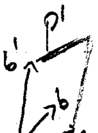
$\Gamma \subset M(2)$ discrete subgroup, no arbitrarily small translations or rotations

$\mathbb{R}^2 \triangleleft M(2) \to O(2) = M(2)/\mathbb{R}^2$ $\mathbb{R}^2$ = translations $t_b(v) = v+b$ $M(2) \ni (A,b)$ $Av+b$

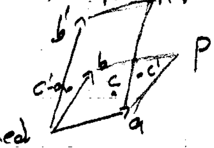
$L = \Gamma \cap \mathbb{R}^2$ the lattice of translation $\bar\Gamma = \Gamma/L \subset O(2)$ a finite subgroup of $O(2)$

Thm There are three possibilities for L: ① $L=0$ ② $L = \mathbb{Z}a$ $a \neq 0$ ③ $L = \mathbb{Z}a + \mathbb{Z}b$, a,b a basis $\mathbb{R}^2$

Pf: ① ✓ ② occurs when $L \subset \mathbb{R}a \subset \mathbb{R}^2$ lies on a line



③ Assume there are independent translations a' and b' in L: $\mathbb{R}a' + \mathbb{R}b' = \mathbb{R}^2$
Following ② we consider the translations in L on the line $\mathbb{R}a'$ and find the shortest one a. Then $L \cap \mathbb{R}a' = \mathbb{Z}a$. Consider the parallelogram spanned by a and b'. Then the intersection $P' \cap L$ where P' is this parallelogram is a finite set. Choose b in this set not on $\mathbb{R}a$ but closest to this line. Then $\mathbb{R}a + \mathbb{R}b = \mathbb{R}^2$.
Claim $\mathbb{Z}a + \mathbb{Z}b = L$. Let P be the parallelogram spanned by a and b. Claim $P \cap L = \{0, a, b, a+b\}$. If $c \in P$ then either $c \in P'$ is closer to $\mathbb{R}a$ than b, or $c - a \in P'$ and is closer to $\mathbb{R}a$ than b, a contradiction.
If c lies on $[b, a+b]$ then $c - b \in \mathbb{R}a$ is smaller than a, also a contradiction. So $P \cap L = \{0, a, b, a+b\}$ as claimed



Now let $v \in L \subset \mathbb{R}^2$. Then $v = xa + yb = ma + nb$ + (remainder of coefficients lie in $[0,1)$)
Then $v - ma - nb \in L \cap P \Rightarrow$ is $\{0, a, b, a+b\} \Rightarrow v \in \mathbb{Z}a + \mathbb{Z}b$.

Prop The image $\bar\Gamma$ of Γ in $O(2)$, which acts on $\mathbb{R}^2$, preserves the subgroup L of translations in Γ.

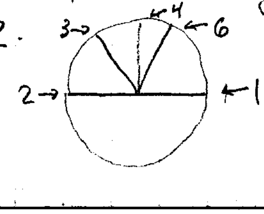
Pf: Let $\bar g \in \bar\Gamma \subset O(2) \hookrightarrow M(2)$. Let $b \in L$, i.e. $t_b \in \Gamma$. We must show $\bar g(b) \in L$, i.e. $t_{\bar g(b)} \in \Gamma$. But $t_{\bar g(b)} = \bar g \circ t_b \circ \bar g^{-1}$ in $M(2)$ and as $\mathbb{R}^2 \triangleleft M(2)$ this is some translation that takes 0 to $\bar g(b)$.

- If $L = 0$ then $\bar\Gamma$ can be any finite subgroup of $O(2)$ = cyclic or dihedral.
- If $L = \mathbb{Z}a$ then any $\bar g \in \bar\Gamma$ must take $\bar g(a) = \pm a$. So $\bar g$ = id or rot by $180°$ or reflection in $\mathbb{R}a$ or reflection in a line $\perp$ to $\mathbb{R}a$. So $\bar\Gamma = \{e\}$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- If $L = \mathbb{Z}a + \mathbb{Z}b$, rotations in $\bar\Gamma$ have order $1, 2, 3, 4$, or 6 $\Rightarrow \#\bar\Gamma \leq 12$. Consider $\mathrm{rot}(\theta) \in \bar\Gamma$

$\mathrm{rot}\,\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Consider the matrix of $\bar g = \mathrm{rot}\,\theta$ with respect to the basis $\{a,b\}$ of $\mathbb{R}^2$. Since $\bar g(L) \subset L$, $\bar g(a) = ma + nb$, $\bar g(b) = pa + qb$ $m,n,p,q \in \mathbb{Z}$

$\bar g = \begin{pmatrix} m & p \\ n & q \end{pmatrix}$. Trace$(\bar g)$: $2\cos\theta = m + q$ is thus an integer between -2 and 2 $\Rightarrow$
$2\cos\theta = -2, -1, 0, 1,$ or 2.

Check $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$, or $\pi \Rightarrow n$ is:



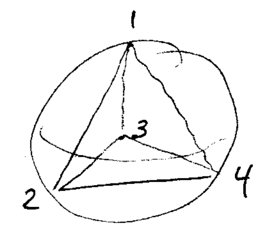
Recall G a group acting on a set S. $O_s = \{s' : g(s) = s'\}$, $G_s = \{g \in G : g(s) = s\} \subset G$

$S = \bigcup_{\text{orbits}} O_s$ If $s' = g(s)$ another point in the orbit then $s = g^{-1}(s')$ and $G_{s'} = g G_s g^{-1}$ is conjugate to G_s. Note for $h \in G_s$, $ghg^{-1}(s') = gh(s) = g(s) = s' \Rightarrow ghg^{-1} \in G_{s'}$.

Hence $G/G_s \xrightarrow{\sim} O_s$ $gG_s \longleftrightarrow s' = g(s)$ as sets

Say G and S are finite. Then $\# S = |S| = \sum_{\text{orbits}} |O_s| = \sum_{\text{orbits}} |G|/|G_s|$ as $O_s \simeq G/G_s$. Well defined because $|G_s| = |G_{s'}|$ for any $s' \in O_s$ by the above computation ($G_s, G_{s'}$ conjugate subgroups).

ex. $G \subset SO_3$ acts on S^2 the two sphere stabalizing a regular tetrahedron $T \subset S^2$. Know $G \subset S_4$ as it permutes the set $\{1,2,3,4\}$ and $\# S_4 = 4! = 24$.



G acts transitively on the four vertices so $|O_s| = 4$ for each $s \in \{1,2,3,4\}$. What is the stabalizer of 1? $G_1 = \{e, \text{rot}(120°) = (234), \text{rot}(240°) = (243)\} \Rightarrow$ $|G| = |O_s| \cdot |G_s| = 4 \times 3 = 12$. In fact, $G = A_4 \triangleleft S_4$ generated by 8 3-cycles (abc).

Application of G acts on $S = G$ by conjugation. $|G| = \sum |G|/|G_s| \leftarrow$ the conjugacy classes.